Examination Introduction to Stochastic Processes
(LNMB/Dutch Master Program)
Monday December 19, 2005, 12.00 - 15.00 hours

This exam consists of 5 problems Use of book is not permitted Motivate your answers!

1. Suppose that print jobs arrive at a network printer with independent and exponentially distributed inter-arrival times with parameter 10 per hour. It takes the printer exactly 6 seconds to print each page. The sizes of the jobs are i.i.d. and have a Poisson distribution with a mean of 2 pages.

   a. What is the probability that precisely 20 jobs arrive between 8.30 hours and 10.30 hours?

   b. What is the probability that a print job arrives while the previous job is not finished, when this previous job consists of 6 pages? (Assume that the printing of this previous job started immediately after its arrival).

   c. What is the expected arrival time of the first job after 12.00 hours, when the previous job arrived at 11.58 hours?

   d. Let $M(t)$ be the number of print jobs consisting of more than 3 pages that arrive in a time interval $(0, t]$. Give an expression for $P(M(t) = m)$.

2. In a simple model, the state of a car dynamo at the beginning of year n is given by a random variable X_n, $(n = 0, 1, 2 \ldots)$, which takes values between 0 and 6. When $X_n = 0$, the dynamo is 'as new', when $X_n = 6$, it is in 'very bad' condition. Depending on the state i of the dynamo at the beginning of a year, it breaks down during this year with probability $i/6$. After a breakdown, the dynamo is completely revised, so that it is as new at the start of the next year. If a dynamo does not break down during a year, the state is increased by one at the end of the year.

   a. Show that $\{X_n, n \geq 0\}$ is a discrete time Markov chain and determine whether the chain is: (i) irreducible, (ii) aperiodic, (iii) transient.

   b. Give the long run probability that the dynamo is 'as new'.

   c. Give the long run probability that the dynamo is 'as new', and was in state 2 the year before, i.e. determine $\lim_{n\to\infty} P(X_{n-1} = 2, X_n = 0)$.

   d. Determine the mean time between two revisions.

3. GSM base stations (antenna's) support mobile telephone conversations. Let us consider such a base station and assume that requests for telephone calls arrive according to a Poisson process with rate 30 per hour, while the length of the calls is exponentially distributed with mean 1 minute. Let $X(t)$ be the number of supported calls at time t. Then $\{X(t), t \geq 0\}$ is a continuous-time Markov chain with state space $\{0, 1, 2, \ldots\}$. Note that when $X(t) = n$, all n calls are in progress simultaneously, so calls are not 'waiting'.

a. Let $n \geq 0$. Give the distribution of the amount of time that the process spends in state n, before making a transition into a different state.

b. Show that the limiting distribution of $\{X(t), t \geq 0\}$ is Poisson with parameter $1/2$.

In reality, base stations can only support a limited number of telephone conversations at the same time. Consider a small base station that can only support 4 conversations (in real stations this capacity is much higher).

c. From question b., find the long-run probability that the station is completely occupied by using a truncation argument. Which property of $\{X(t), t \geq 0\}$ do you use?

4. A machine is subject to shocks that occur according to a Poisson process with rate λ per hour. After each shock, the machine breaks down and goes into repair for a stochastic time which is uniformly distributed on $[0, T]$. Shocks that occur during repair have no effect. After repair, the machine works properly again. Let $N(t)$ denote the number of breakdowns in the interval $(0, t]$.

   a. Assuming that a breakdown occurs at time 0, is the process $\{N(t), t \geq 0\}$ a renewal process? Why (not)?

   b. On average, how frequently does the machine break down in the long run?

   c. What is the long run probability that the system is in repair?

5. Consider a normal renewal process with strictly positive life times X_i, mean life time $\mu = EX_1$, life time LST $\phi(s) = Ee^{-sX_1}$, and renewal function $m(t) = EN(t)$.

   a. Let $S_N(t) = \sum_{i=1}^{N(t)} X_i$ be the time of the last renewal before (or at) time t, and $S_{N(t)+1} = \sum_{i=1}^{N(t)+1} X_i$ be the time of the next renewal after time t. Note that $X_{N(t)+1} = S_{N(t)+1} - S_{N(t)}$ is the length of the renewal interval containing t. It is known that $ES_{N(t)+1} = \mu[m(t) + 1]$. Can we also argue that $ES_{N(t)} = \mu m(t)$? Explain why or why not.

   b. The limiting distribution of the excess time is given by $P(Y \leq x) = \mu^{-1} \int_0^x P(X > y)\,dy$. Show that the corresponding Laplace-Stieltjes Transform (LST) is given by $Ee^{-sY} = (1 - \phi(s))/s\mu$.

Standard:

1				2				3			4			5		total
a	b	c	d	a	b	c	d	a	b	c	a	b	c	a	b	
1	2	2	3	2	2	2	2	2	3	3	2	2	2	3	3	+ 4 = 40