Algebraic Number Theory — January 17, 2019

Problem 1.

For the field $K = \mathbb{Q}(\sqrt{61})$ give the ring of integers $\mathcal{O}_K$, the class group of $\mathcal{O}_K$, and a unit of infinite order in $\mathcal{O}_K$.

Problem 2.

Let ζ be a primitive 7th root of unity, let $\eta = \zeta + \zeta^{-1}$, and let $K = \mathbb{Q}(\eta)$.

(a) Find the minimal polynomial of η.

(b) Prove that the discriminant of $\mathbb{Z}[\eta]$ is 49.

(c) Show that $\mathcal{O}_K = \mathbb{Z}[\eta]$.

(d) Show that the class group of K is trivial.

Problem 3.

Let R be the ring $\mathbb{Z}[\sqrt{2}, \sqrt{3}]$.

(a) Show that R has a unique prime ideal $\mathfrak{p}$ such that the index $[R : \mathfrak{p}]$ is a power of 2.

(b) Show that $[R : \mathfrak{p}] = 2$ and that $[\mathfrak{p} : \mathfrak{p}^2]$=4.

(c) Show that $\mathfrak{p}$ is a singular prime of R.

(d) Give an element of the integral closure of R that does not lie in R.

Problem 4.

(a) Show that the polynomial $f = X^3 + 7$ is irreducible in $\mathbb{Q}[X]$.

(b) Let K be the number field $\mathbb{Q}[X]/(f)$. Find the ring of integers O_K of K.

(c) Determine the class group of K.