Please write clearly and give full proofs of all your claims. This exam is open book, which means you are allowed to use the course material "Number Rings" written by P. Stevenhagen, as well as any notes taken during lectures. In your solutions, you may refer to numbered results and examples contained "Number Rings". You are not permitted to use any other tools.

Problem 1. Let $\mathcal{O}_K$ be the ring of integers of the quadratic field $K = \mathbb{Q}(\sqrt{-26})$. Compute the unit group $\mathcal{O}_K^\times$ and prove that the class group $\mathrm{Cl}(\mathcal{O}_K)$ is cyclic of order 6.

Problem 2. Suppose $g > 1$ is an integer, and $n > 1$ is an odd integer, such that

$$d := n^g - 1$$

is squarefree, i.e. is not divisible by the square of any prime number.

- Prove that $(1 + \sqrt{-d}) = I^g$ for some ideal I in the ring of integers of $\mathbb{Q}(\sqrt{-d})$.
- Prove that the ideal class group of $\mathbb{Q}(\sqrt{-d})$ has an element of order g.

(For example, when $g = n = 3$ we obtain $d = 26$, see Problem 1.)

Problem 3. Find an example of a number field K whose ring of integers $\mathcal{O}_K$ has precisely two primes above 2 and precisely three primes above 3. Prove your answer.

Problem 4. Consider the number ring $R = \mathbb{Z}[\sqrt{8}]$.

(1) Find the unit group $R^\times$.

(2) Find three different positive solutions for $x, y \in \mathbb{Z}_{>0}$ of the equation

$$17 = x^2 - 8y^2.$$

How many solutions are there? Prove your answer.

(3) Find all singular primes of R and prove that $\mathrm{Pic}(R) = 1$.

(4) Prove that a prime p satisfies $p \equiv 1 \pmod 8$ if and only if it is of the form

$$p = x^2 - 8y^2, \qquad x, y \in \mathbb{Z}.$$

(Hint: Use Kummer–Dedekind. You may use without proof the fact that any odd prime p satisfies $p \equiv \pm 1 \pmod 8$ if and only if 2 is a square modulo p.)