Algebraic Number Theory, January 16, 2020, 10:00 – 13:00

Problem 1. Let $\mathcal{O}$ be the ring of integers of the the quadratic field $K = \mathbf{Q}(\sqrt{-71})$. Compute the unit group $\mathcal{O}^*$ and the order of the class group $Cl(\mathcal{O})$.

Problem 2. Find all solutions in integers of the equation $x^2 = y^3 - 13$.

Problem 3. We call a positive integer $y \in \mathbf{Z}_{>0}$ *peculiar* if the difference between the consecutive cubes y^3 and $(y+1)^3$ is a square.
Example: 7 is peculiar, as $7^3 = 343$ and $8^3 = 512$ have difference $512 - 343 = 169 = 13^2$.
(a) Find the next peculiar number.
(b) Show that there are infinitely many peculiar numbers.
(c) Show that the quotient of 2 consecutive peculiar integers tends to $7 + 4\sqrt{3} \approx 13.9282$.

Problem 4. Let K_1 and K_2 be the cubic number fields obtained by adjoining a root of $f_1 = X^3 + 15X - 15$ and $f_2 = X^3 - 15X + 35$, respectively.
(a) Find the discriminants of K_1 and K_2.
(b) Are K_1 and K_2 isomorphic number fields?