Field theory in condensed matter

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Final exam (43 points) 28 → 31
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- Use a separate sheet for every exercise.
- Write your name on each sheet, on the first sheet also your student ID.
- Write clearly, unreadable work cannot be corrected.
- Give the motivation, explanation and calculations.
- Do not spend a large amount of time on finding (small) calculation errors. If you suspect you have made such an error, point it out in words.
- No exercise requires a long calculation
- All the questions are preceded by bullet points!!!

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1. The xxz-chain (27 points)

The Hamiltonian of the xxz-chain reads

$$H = -J\sum_i \left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z\right) - \vec{h}\sum_i \vec{\sigma}_i \tag{1}$$

where $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ with σ^x, σ^y, and σ^z being the Pauli matrices and i denotes the spatial position of the spin.

- In the limit $\Delta = 1$ and $\vec{h} = 0$ (for the moment we consider a d dimensional version of the model with nearest neighbor interaction) this model is called the Heisenberg model and it has a continuous spin symmetry. Explain in maximally three sentences what the Mermin-Wagner theorem states about the possibility of symmetry breaking and comment specifically on the role of temperature and dimensionality. (2 points) 1

We are going back to the one dimensional model, now. In the limit $\Delta \gg 1$ (you might assume $\Delta \to \infty$) and $\vec{h} = 0$, the model in Eq. (1) is the classical Ising model.

- What is/are the ground state/s in that situation and comment on the degeneracy (2 points)? Does this model have a phase transition at finite temperature (1 point)? 2
- Upon switching on a transverse field, *i.e.*, $\vec{h} = (h_0, 0, 0)^T$ with $h_0 \neq 0$, one obtains a transverse field or quantum Ising model. This model has a quantum phase transition at zero temperature. Sketch its phase diagram and characterize the phases through an appropriate order parameter or alternatively the behavior of a correlation function (4 points). 3

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The xx-chain ($\Delta = 0$) with $\vec{h} = (0, 0, h_0)^T$ belongs to the class of problems that can be solved exactly via Jordan-Wigner transformation. The transformation is given by $\sigma_i^x = \Pi_{j<i}\left(1 - 2c_j^\dagger c_j\right)\left(c_i + c_i^\dagger\right)$, $\sigma_i^y = i\Pi_{j<i}\left(1 - 2c_j^\dagger c_j\right)\left(c_i - c_i^\dagger\right)$, and $\sigma_i^z = \left(1 - 2c_i^\dagger c_i\right)$, where the c_i obey the standard anticommutation relations, *i.e.*, $\{c_i, c_j^\dagger\} = \delta_{i,j}$ and $\{c_i^\dagger, c_j^\dagger\} = \{c_i, c_j\} = 0$.

- Show that the model can be rewritten in terms of fermionic operators as
$$H = -2J\sum_i\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) + 2h_0\sum_i\left(c_i^\dagger c_i - \frac{1}{2}\right). \tag{2}$$
(4 points) 4
- In a ring geometry the above model, Eq. (2), can be solved using Fourier transformation. Because of the string of operators in the Jordan-Wigner transformation, it splits into two parity (number of electron) sectors, even and odd, which translates to periodic and antiperiodic boundary conditions. Use Fourier transformation to determine the energy spectrum and give the quantization condition on the momentum in the two cases of periodic and antiperiodic boundary conditions, respectively (4 points).
- Show that switching on $\Delta \neq 0$ drives the system away from a model of free fermions by leading to an interaction term of the form $-4J\Delta\sum_i\left(c_i^\dagger c_i - \frac{1}{2}\right)\left(c_{i+1}^\dagger c_{i+1} - \frac{1}{2}\right)$. (1 point) 1

One can derive a low-energy field theory for the above problem. In the following discussion we will need to introduce two sets of fermionic fields, ψ_L and ψ_R which denote left and right movers, respectively. They obey the standard commutation relations for fermionic fields and are a result of zooming in on the vicinity of the chemical potential which we assume to be given by $h_0 = 0$. The free action ($\Delta = 0$) of the problem in Euclidean space time is given by two one-dimensional Dirac fermions with opposing chiralities and reads

$$S_0 = \int dx \int d\tau (\bar{\psi}_R(x,\tau), \bar{\psi}_L(x,\tau)) \begin{pmatrix} \partial_\tau + iv_F\partial_x & 0 \\ 0 & \partial_\tau - iv_F\partial_x \end{pmatrix} \begin{pmatrix} \psi_R(x,\tau) \\ \psi_L(x,\tau) \end{pmatrix}. \tag{3}$$

This field theory describes the physics of the low-energy fixed point of the xx chain.

- Determine the scaling dimension of the fields $\psi_{R/L}$ (if you fail to do so you might continue with assuming it is given by δ) (2 points). 2

We now go away from the xx fixed point by adding perturbations

$$\begin{aligned}
\delta S_1 &= m\int dx \int d\tau (\bar{\psi}_R(x,\tau), \bar{\psi}_L(x,\tau)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_R(x,\tau) \\ \psi_L(x,\tau) \end{pmatrix} \\
\delta S_2 &= \alpha\int dx \int d\tau \bar{\psi}_R(x,\tau)\psi_R(x,\tau)\bar{\psi}_L(x,\tau)\psi_L(x,\tau) \\
\delta S_3 &= \beta\int dx \int d\tau \bar{\psi}_R(x,\tau)\psi_R(x,\tau)\partial_x^2\bar{\psi}_L(x,\tau)\psi_L(x,\tau)\,,
\end{aligned} \tag{4}$$

where α and β are directly proportional to Δ.

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- What are the scaling dimensions of m, α, and β? (6 points) ✓
- Are they relevant, irrelevant, or marginal, respectively? (1 point) (if you fail to answer the preceding question give a general statement) ✓

(hint: do not get distracted by the matrix structure of the problem)

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2. Fock-Darwin spectrum (16 points)

A special setup in which one can study effects of quantum mechanics in two dimensions are so-called large dots. We can model them by adding a parabolic confining potential of the form $V(x,y) = \frac{1}{2}m\omega_0^2(x^2+y^2)$ to a two dimensional electron gas. The Hamiltonian under consideration reads

$$H = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega_0^2\left(x^2+y^2\right) \tag{5}$$

where $\vec{p} = (p_x, p_y)$. Adding a magnetic field can be accounted for by introducing the canonical momentum, $\vec{\Pi} = \vec{p} + e\vec{A}$ ($\vec{p} = -i\vec{\partial}$). We want to add a perpendicular magnetic field, $\vec{B} = (0,0,B)^T$.

- Two of the most popular gauge choices are the Landau gauge, $A_L = -B(y,0,0)^T$ (or $A_L' = B(0,x,0)^T$), and the symmetric gauge, $A_S = -\frac{B}{2}(y,-x,0)$. Give an argument why in our setup we should use the symmetric gauge for technical reasons (2 points). ~ 2 1

In the following we use an algebraic approach based on raising and lowering operators combined with a Bogoliubov transformation to solve the problem. In the following you can use the cyclotron frequency $\omega_c = \frac{eB}{m}$ for a more compact notation.

- Show that one can write the Hamiltonian in the form $H = \frac{p_x^2+p_y^2}{2m} + \frac{1}{2}m\omega_{eff}(x^2+y^2) + \frac{\omega_c}{2}(xp_y - yp_x)$ and determine ω_{eff} (4 points). Furthermore, show that $H_0 = \frac{p_x^2+p_y^2}{2m} + \frac{1}{2}m\omega_{eff}(x^2+y^2)$ commutes with $H_1 = \frac{\omega_c}{2}(xp_y - yp_x)$ (reminder: $[x,p_x] = i$; it is enough to show it for one direction, for instance x, explicitly) (2 points). 4 2

We can introduce raising and lowering operators which solve H_0 according to

$$\begin{aligned} a_x &= \sqrt{\frac{m\omega_{eff}}{2}}\left(x + i\frac{1}{m\omega_{eff}}p_x\right) \; a_x^\dagger = \sqrt{\frac{m\omega_{eff}}{2}}\left(x - i\frac{1}{m\omega_{eff}}p_x\right), \\ a_y &= \sqrt{\frac{m\omega_{eff}}{2}}\left(y + i\frac{1}{m\omega_{eff}}p_y\right) \; a_y^\dagger = \sqrt{\frac{m\omega_{eff}}{2}}\left(y - i\frac{1}{m\omega_{eff}}p_y\right). \end{aligned} \tag{6}$$

They satisfy the canonical commutation relations $[a_x, a_x^\dagger] = [a_y, a_y^\dagger] = 1$ and $[a_x, a_y^\dagger] = [a_y, a_x^\dagger] = 0$. This allows to introduce number states $|n_x, n_y\rangle$ where the operators act like $a_x|n_x,n_y\rangle = \sqrt{n_x}|n_x-1,n_y\rangle$, $a_x^\dagger|n_x,n_y\rangle = \sqrt{n_x+1}|n_x+1,n_y\rangle$, $a_y|n_x,n_y\rangle = \sqrt{n_y}|n_x,n_y-1\rangle$, and $a_y^\dagger|n_x,n_y\rangle = \sqrt{n_y+1}|n_x,n_y+1\rangle$.

- Show that in terms of these operators $H_0 = \omega_{eff}(a_x^\dagger a_x + a_y^\dagger a_y + 1)$ and $H_1 = -i\frac{\omega_c}{2}\left(a_x^\dagger a_y - a_y^\dagger a_x\right)$. (2 points) ✓ 2

In order to solve the Hamiltonian we need to perform a change of basis according to $b_1 = \frac{1}{\sqrt{2}}(ia_x + a_y)$, $b_1^\dagger = \frac{1}{\sqrt{2}}\underline{\left(-ia_x^\dagger + a_y^\dagger\right)}$, $b_2 = \frac{1}{\sqrt{2}}(-ia_x + a_y)$, and $b_2^\dagger = \frac{1}{\sqrt{2}}\underline{\left(ia_x^\dagger + a_y^\dagger\right)}$. These operators again fulfil canonical commutation relations, *i.e.*, $[b_1, b_1^\dagger] = [b_2, b_2^\dagger] = 1$ and $[b_1, b_2^\dagger] = [b_2, b_1^\dagger] = 0$, meaning we can again introduce number states according to $|n_1, n_2\rangle$ with $b_1|n_1,n_2\rangle = \sqrt{n_1}|n_1-1,n_2\rangle$, $b_1^\dagger|n_1,n_2\rangle = \sqrt{n_1+1}|n_1+1,n_2\rangle$, $b_2|n_1,n_2\rangle = \sqrt{n_2}|n_1,n_2-1\rangle$, and $b_2^\dagger|n_1,n_2\rangle = \sqrt{n_2+1}|n_1,n_2+1\rangle$. 4

- Show that the full Hamiltonian H is diagonal in these operators and give its spectrum. (6 points)