Differential Topology - Midterm Examination (November 1st, 2012)

1. *Please...*
   (a) *make sure your name and student number are written on every sheet of paper that you hand in;*
   (b) *start each exercise on a new sheet of paper and number each sheet.*
2. *All results from the lectures and the exercises can be taken for granted, but must be stated when used.*

Exercise 1 (4 points)**.** A *topological group* is a group G together with a topology, such that multiplication and inversion are continuous maps. For instance, $(\mathbb{R}, +)$ is a topological group.

1. Prove that the map

$$\begin{array}{rcl} \times : \mathcal{C}^\infty_W(\mathbb{R},\mathbb{R}) \times \mathcal{C}^\infty_W(\mathbb{R},\mathbb{R}) & \to & \mathcal{C}^\infty_W(\mathbb{R}^2,\mathbb{R}^2), \\ (f,g) & \mapsto & (f \times g)(x,y) := (f(x), g(y)) \end{array}$$

   is continuous.

2. Use point 1. to conclude that $\mathcal{C}^\infty_W(\mathbb{R},\mathbb{R})$ is a topological group with respect to the usual addition of functions.

3. Show that the usual multiplication by scalars

$$\mathbb{R} \times \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R}), \qquad (c,f) \mapsto c \cdot f,$$

   is not continuous with respect to the strong topology.

Exercise 2 (6 points)**.** Recall that two continuous maps f, $g : S^1 \to S^1$ are homotopic if there is a continuous map $H : S^1 \times [0,1] \to S^1$ such that $H|_{S^1\times\{0\}} = f$ and $H|_{S^1\times\{1\}} = g$. Take the following facts for granted:

- Being homotopic is an equivalence relation.
- $\pm\mathrm{id}$ is not homotopic to any constant map.
- If $f : S^1 \to S^1$ is not surjective, it is homotopic to a constant map.
- If $f, g : S^1 \to S^1$ satisfy $f(x) \neq -g(x)$ for all $x \in S^1$, then

  $$H(t,x) := (tf(x) + (1-t)g(x))/||tf(x) + (1-t)g(x)||$$

  is a homotopy between f and g.

1. We denote the standard coordinate on S^1, which runs from 0 to 2π, by θ. Given a smooth immersion $u : S^1 \to \mathbb{R}^2$, we define $W_u : S^1 \to S^1$ by

   $$W_u(\theta) := \frac{du/d\theta}{||du/d\theta||}.$$

   Prove that

   $$W : \mathsf{Imm}^\infty_S(S^1, \mathbb{R}^2) \to \mathcal{C}^\infty_S(S^1, S^1), \qquad u \mapsto W_u$$

   is continuous.

2. Consider the figure eight $\infty \subset \mathbb{R}^2$, parametrized by $u_\infty(\theta) = (\cos\theta, \sin 2\theta)$.
   (a) Prove that u_∞ is an immersion.
   (b) Prove that the image of W_{u_∞} does not contain $(0,-1) \in S^1$.

3. Given $f \in \mathcal{C}^\infty_S(S^1, S^1)$, describe a neighborhood $\mathcal{U}$ of f such that if $g \in \mathcal{U}$, then $g(x) \neq -f(x)$ holds for all $x \in S^1$.

4. Use the previous points and the fact that

   "If $u : S^1 \to \mathbb{R}^2$ is an embedding, then W_u is homotopic to $\pm$identity.",

   which you can take for granted, to prove that $\mathsf{Emb}^\infty(S^1, \mathbb{R}^2)$ is not dense in $\mathcal{C}^\infty_S(S^1, \mathbb{R}^2)$.