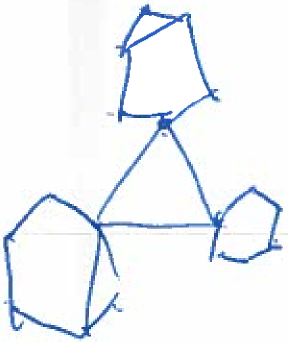
10= 21'



Methods and Models in Complex Systems
BETA-B2-CS – Exam

November 6, 2019

(Time: 3 hours. Please motivate your answers and simplify your results where possible.)

1. The cable infrastructure in the Netherlands that provides our Internet services consists of rings that are connected with each other. The main ring is a fiber-optic cable with n_f nodes called *distribution points* and n_f links between these nodes. Each distribution point has its own copper cable ring with n_c nodes representing the homes that it connects. The distribution point is part of that ring as well, which thus has $n_c + 1$ nodes and $n_c + 1$ links. Assume for convenience that n_f and n_c are odd.

   (a) [1 pt] Why would the network be built from rings?

   (b) [2 pt] Draw the corresponding graph for $n_f = 3$ and $n_c = 5$.

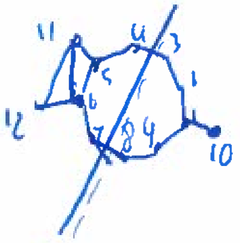
   (c) [1 pt] How many edges does the graph possess for general n_f and n_c?

   (d) [2 pt] What is the diameter of the graph for general n_f and n_c?

   (e) [1 pt] What is the diameter for general n_f and n_c if we would cut one of the fiber-optic links?

   (f) [2 pt] Determine the betweenness centrality of a distribution point, for $n_f = 3$ and $n_c = 5$.

   (g) [1 pt] Determine the betweenness centrality of a home farthest away from its distribution point, for $n_f = 3$ and $n_c = 5$.



2. Given is a social network of 12 persons, defined as follows. There is a ring of 9 persons, numbered 1–9, where i and $i+1$ are linked for $i = 1, 2, \ldots 8$, and person 9 is linked to 1. There are three more persons (10,11, 12) who are not in the ring. Person 10 is connected to 1, person 11 to 5 and 6, person 12 to 6 and 11. Furthermore, we have some shortcuts in the ring: 1 is connected to 7, and 2 is connected to 8, and 3 is connected to 5 and 6. The graph has been partitioned into two sets: $V_1 = \{1, 2, 3, 8, 9, 10\}$ and $V_2 = \{4, 5, 6, 7, 11, 12\}$.

   (a) [2 pt] Draw the graph and draw a splitting line that separates the two parts, putting vertices of V_1 on side and those of V_2 on the other.

   (b) [2 pt] What is the edge cut of this partitioning?

   (c) [2 pt] The modularity M of a clustered network can be computed by

   $$M = \sum_{c=1}^{n_c} \left[\frac{L_c}{L} - \left(\frac{k_c}{2L} \right)^2 \right].$$

   Here, n_c is the number of clusters (communities), L is the total number of links in the network, L_c is the number of links in community c, and k_c is the sum of the degrees of the nodes in community c.

   What is the modularity of the clustering into two clusters based on the given partitioning into two parts? (Only one decimal digit accuracy is required, so you may round off where needed.)

   (d) [3 pt] Give a Kernighan–Lin swap that reduces the edge cut but retains a perfect balance between the two parts, each keeping 6 persons. What is the new edge cut? Try to reduce the edge cut as much as possible, still keeping balance.

   (e) [1 pt] Compute the modularity of the corresponding new clustering.

4. Three airports: Amsterdam, Milan and Paris are connected with multiple flights as shown. Each arch at the map denotes one flight connection. You can see that the number of flights is different between different cities. This is because the airport network have evolved to accommodate the uneven passenger flow. To estimate the number of customers, the hotel management applies the following Markovian model: every day a passenger X takes a different flight and stays in the city over night. This person cannot use other means of transport, so, for example, if X is in Paris, the next day X will be in Amsterdam or Milan. X is equally likely to take each flight.

(a) [3 pt] Write down the transition probability matrix for X.

(b) [3 pt] Suppose X is in Paris, what is the probability X will again be in Paris in two days?

(c) [4 pt] Given passenger X is a representative model for all passengers, what fractions of overnight stays

$$0 \le f_{\mathrm{Pa}}, f_{\mathrm{Mi}}, f_{\mathrm{Am}} \le 1, \; f_{\mathrm{Pa}} + f_{\mathrm{Mi}} + f_{\mathrm{Am}} = 1$$

do passengers spend at each city? Note that the symmetry implies that these values should be the same for Paris and Milan,

$$f_{\mathrm{Pa}} = f_{\mathrm{Mi}}.$$

3. Some infections, for example those from the common cold and influenza, do not confer any long lasting immunity. Such infections do not give immunisation upon recovery from infection, and individuals become susceptible again. Consider a compartment model in which the population is split into two groups: S (susceptible) and I (infected). The concentrations of S and I individuals in time are denoted by $s(t)$ and $x(t)$ respectively satisfying $s(t) + x(t) = 1$. Consider an infection process:

$$S + I \to 2I$$

having rate $\beta \geq 0$, and a recovery process

$$I \to S$$

with rate $\delta \geq 0$, $\delta \neq \beta$.

(a) [1 pt] Formulate the system of two ordinary differential equations for $s(t)$ and $x(t)$.

(b) [1 pt] Show that this system is equivalent to one differential equation for $x(t)$, write down this equation.

(c) [4 pt] Find all fixed points of the form (x*,s*), classify their stability depending on the parameters δ and β.

(d) [2 pt] Show that when the recovery rate $\delta = 0$ then model can be reduced to the logistic equation. Give an example of a system outside of epidemiology where the logistic equation is useful.

(e) [2 pt] Suppose the individuals from population S attain an immediate and long lasting immunity upon recovery. What alterations should be made to the model to account for such a modification? Formulate the modified model as a system of ODEs.