Mastermath	Commutative algebra resit	14:00-17:00 6/3/2018

- Write your name, university, and student number on every sheet you hand in.
- You may use a printout of Altman-Kleiman's book *A term of commutative algebra.*
- Motivate all your answers.
- If you cannot do a part of a question, you may still use its conclusion later on.

(1) In this problem, X, Y and Z are variables, and x, y and z their images in quotient rings.
  (a) Let $\varphi : \mathbb{R}[X] \to \mathbb{R}[Y]$ be the homomorphism of $\mathbb{R}$-algebras given by mapping X to $Y^3 - 1$, and let $\varphi^* : \operatorname{Spec}(\mathbb{R}[Y]) \to \operatorname{Spec}(\mathbb{R}[X])$ be the induced map.
    (i) Find the image $\varphi^*(\langle Y \rangle)$ in the form $\langle f(X) \rangle$ for a suitable $f(X)$ in $\mathbb{R}[X]$.
    (ii) Find the elements in the fibre $(\varphi^*)^{-1}(\langle X \rangle)$, each in the form $\langle g(Y) \rangle$ for a suitable $g(Y)$ in $\mathbb{R}[Y]$.
  (b) (i) Let k be a field, and $S = k[X, Y, Z]/\langle XYZ - 1 \rangle$. Show that the morphism $k[X, Y] \to S$ of k-algebras mapping X to x and Y to y is injective.
    (ii) Show that S is *not* an integral extension of the image R of the morphism in (i).
    (iii) Find a k-subalgebra R' of S such that S is an integral extension of R' and R' is isomorphic to $k[X, Y]$ as k-algebra.

(2) The following table lists three rings R, where k is a field, and X, Y are variables.

	$R = \mathbb{Z}$	$R = k[X, Y]/\langle XY \rangle$	$R = k[X, Y]/\langle X^2, XY, Y^2 \rangle$
Noetherian			
Artinian		(b)	
Reduced			

  (a) Fill in each box in the table with T or F, according to the property in the row being true or false for the ring R in the column. *Grading: 1 point for each correct answer, -0.5 points for each incorrect answer, 0 points for blank box. Minimum score 0.*
  (b) Prove your answer in the box marked (b).

(3) Let $R = \mathbb{C}[X, Y]$, and define R-modules by $M_1 = R/\langle X^2, XY \rangle$, $M_2 = R[S]/\langle XS - 1 \rangle$, and $M = M_1 \oplus M_2$. Recall that the maximal ideals of R are of the form $\langle X - a, Y - b \rangle$ for $a, b \in \mathbb{C}$, which you may use without proof.
  (a) For which maximal ideals $\mathfrak{m}$ of R is the natural map $M \to M_{\mathfrak{m}}$ injective?
  (b) For which maximal ideals $\mathfrak{m}$ of R is $M_{\mathfrak{m}}$ flat as an R-module?

(4) Let R be a Noetherian ring and M a finitely generated R-module.
  (a) Show that $\operatorname{Supp}(M)$ (with the topology induced from $\operatorname{Spec}(R)$) is Hausdorff if and only if it consists of a finite set of maximal ideals of R.
  (b) Assume $\operatorname{Supp}(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$, where the $\mathfrak{m}_i$ are distinct maximal ideals of R, and let $I = \prod_{i=1}^n \mathfrak{m}_i$. Show that there exists a positive integer t such that the natural map $M \to M \otimes_R R/I^t$ is an isomorphism of R-modules.
  (c) Show that $M = M_1 \oplus \cdots \oplus M_n$ where M_i is a submodule with $\operatorname{Supp}(M_i) = \{\mathfrak{m}_i\}$.
  (d) Show that if $M = N_1 \oplus \cdots \oplus N_n$ with each N_i a submodule with $\operatorname{Supp}(N_i) \subseteq \{\mathfrak{m}_i\}$, then $N_i = M_i$ for each $i = 1, \ldots, n$. (Hint: one way to do this is to write down a suitable element y in R that annihilates $M_2, \ldots, M_n$ and $N_2, \ldots, N_n$.)

Points below; maximum score: 90; exam grade: score/10+1			
1a: 4 + 7 1b: 4 + 7 + 7	2a: 9 2b: 6	3a: 9 3b: 8	4a: 7 4b: 5 4c: 9 4d: 8