Mastermath	**Commutative algebra exam**	**14:00-17:00 30/1/2018**

- Write your name, university, and student number on every sheet you hand in.
- You may use a printout of Altman-Kleiman's book *A term of commutative algebra*.
- Motivate all your answers.
- If you cannot do a part of a question, you may still use its conclusion later on.

(1) The following four parts can be done entirely independently.

(a) (i) Show the rule $a \otimes b \mapsto ab$ gives a well-defined map of $\mathbb{Z}$-modules $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$.

(ii) Show that the map in (i) is an isomorphism of $\mathbb{Z}$-modules.

(b) Let p, q be distinct prime numbers. Show that $(\mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/q\mathbb{Z}) = 0$ as $\mathbb{Z}$-modules.

(c) Let $R = \mathbb{Z}[X]/\langle 2X - 1 \rangle$. Show that R is *not* an integral extension of $\mathbb{Z}$.

(2) Let k be a field. At the top of the following table, two rings R, each with an R-algebra A, are listed.

	$R = k$, $A = k[X,Y]/\langle XY \rangle$	$R = k[X,Y]$, $A = R/\langle XY \rangle$
A is a finitely generated R-algebra		
A is a finitely generated R-module		
A is a flat R-module		(b)

(a) Fill in each box in the table with T or F, according to whether or not the given property is true for the given ring R, and R-algebra A (sometimes viewed as R-module) in that column. **You do not need to justify your answers to this part.** *Grading: 2 points for each correct answer. -1 points for each incorrect answer. 0 points for blank box. Minimum score 0.*

(b) Prove your answer in the box marked (b).

(3) Let k be a field, $R = k[X,Y]$, I the ideal $\langle X^2Y \rangle$ of R, and M the R-module R/I.

(a) Show that the support of I as an R-module is the whole of $\operatorname{Spec} R$.

(b) Show that there are exactly two associated primes of M, namely $\langle X \rangle$ and $\langle Y \rangle$. For each associated prime $\mathfrak{p}$, give an element $m \in M$ such that $\mathfrak{p} = \operatorname{Ann}_R(m)$.

(c) List the minimal primes in the support of M.

(d) Write down a minimal primary decomposition of I as a submodule of R.

(4) Let k be a field, and A a non-zero finitely generated k-algebra of Krull dimension d. For $n \geq 0$, let $P_n = k[X_1, \ldots, X_n]$ be the polynomial algebra on variables $X_1, \ldots, X_n$ with coefficients in k. We shall prove the following statement:

there exists an *injective* k-algebra homomorphism $P_n \to A$ if and only if $n \leq d$.

In the process, we need (a)(i) and (c), **which can be done entirely independently.**

(a) (i) Show that $\langle 0 \rangle \subsetneq \langle X_n \rangle \subsetneq \cdots \subsetneq \langle X_2, \ldots, X_n \rangle \subsetneq \langle X_1, \ldots, X_n \rangle$ is a maximal chain in $\operatorname{Spec}(P_n)$.

(ii) Use (i) to show that d is finite. (Hint: show that $d = \nu$ in (15.1).)

(b) Prove that there exists an injective k-algebra homomorphism $\varphi : P_n \to A$ if $n \leq d$.

(c) Let R be a domain, S a non-zero Noetherian ring, and $\varphi : R \to S$ an injective ring homomorphism. Prove that there exists a minimal prime ideal Q of S such that the composition $R \xrightarrow{\varphi} S \to S/Q$ is injective, where $S \to S/Q$ is the quotient map.

(d) Let $\varphi : P_n \to A$ be an injective k-algebra homomorphism. Using (c), or otherwise, prove that $n \leq d$. (Hint: you may use without proof that if $k \subseteq K \subseteq L$ are fields, then $\operatorname{tr.deg}_k(K) \leq \operatorname{tr.deg}_k(L)$.)

Points below; maximum score: 90; exam grade: score/10+1			
1a: 3 + 5 1b: 5 1c: 5	2a: 12 2b:6	3a: 4 3b: 9 3c: 4 3d: 6	4a: 7+4 4b: 4 4c: 8 4d: 8