Mastermath	Commutative algebra practice exam	2017-2018 (three hours)

- Write your name, university, and student number on every sheet you hand in.
- You may use a printout of Altman-Kleiman's book *A term of commutative algebra.*
- Motivate all your answers.
- If you cannot do a part of a question, you may still use its conclusion later on.

(1) (a) Let k be a field, x and y variables, $R = k[[x]] \times k[y]$, and $f = (x, 0)$. Show that there is exactly one prime ideal P of R with $P \cap \{1, f, f^2, f^3, \dots\} = \emptyset$, and that P is not a maximal ideal.

(b) Let k be a field, A a finitely generated k-algebra. Show that for f in A *not* nilpotent there exists a maximal ideal P of A with $P \cap \{1, f, f^2, f^3, \dots\} = \emptyset$.

(2) At the top of the following table, three rings R, each with an R-module M, are listed.

	$R = \mathbb{Z}$, $M = \mathbb{Q}$	$R = k$, $M = k[X]$	$R = k[X]$, $M = k$ where X acts as 0
flat			
faithfully flat	(b)		
finitely generated			
finitely presented		(c)	

(a) Fill in each box in the table with T or F, according to whether or not the given property is true for the given R-module M in that column. *Grading: 1 point for each correct answer. -0.5 points for each incorrect answer. 0 points for blank box. Minimum score 0.*

(b) Prove your answer in the box marked (b).

(c) Prove your answer in the box marked (c).

(3) Let $\varphi : R \to R'$ be a ring homomorphism, and $\varphi^* : \operatorname{Spec}(R') \to \operatorname{Spec}(R)$ the induced map. Assume that φ^* maps open sets to open sets.

(a) Show that if Q' is in $\operatorname{Spec}(R')$, P and Q are in $\operatorname{Spec}(R)$, Q' maps to Q, and $P \subseteq Q$, then P is in the image of φ^*.

By (a) we know there exists P' in $\operatorname{Spec}(R')$ with P' lying over P. We want to show that there exists such a P' with $P' \subseteq Q'$, i.e., that *going down* holds.

We proceed by contradiction, so assume that for all P' lying over P we have $P' \not\subseteq Q'$. In order to lighten notation, we let $K = \operatorname{Frac}(R/P)$. Also, $R'_{Q'}$ and R'_f below are viewed as R-modules under the natural compositions $R \to R' \to R'_{Q'}$ and $R \to R' \to R'_f$.

(b) Explain why $R'_{Q'} \otimes_R K = 0$. (Hint: consider the natural homomorphism $R' \to R'_{Q'}$.)

(c) Prove that for every f in $R' \setminus Q'$ we have $R'_f \otimes_R K \neq 0$. (Hint: the image of $\operatorname{Spec}(R'_f) \to \operatorname{Spec}(R')$ is open.)

(d) Explain why (b) and (c) are in contradiction (which finishes the proof).

(4) **In this problem, (b), (c) and (d) are independent of each other.**
Let $R \neq \{0\}$ be a Noetherian ring, with minimal prime ideals $P_1, \dots, P_r$ ($r \geq 1$). Let

$$Z = \{a \text{ in } R \text{ such that multiplication by } a \text{ on } R \text{ is not injective}\}$$

be the set of zero-divisors of R.

(a) Show that $P_1 \cup \dots \cup P_r \subseteq Z$.

(b) Prove that if $\operatorname{nil}(R) = \{0\}$ then equality holds in (a). (Hint: you may want to consider a ring homomorphism to $\prod_{i=1}^{r} R/P_i$.)

(c) Let K be a field. Show that for $R = K[x, y]/(x^2)$ equality holds in (a), but that for $R = K[x, y]/(x^2, xy)$ equality does not hold.

(d) Let $b \in R \setminus Z$ and let S be the quotient ring R/bR. Viewing $\mathrm{Spec}(S)$ as $\mathcal{V}_{\mathrm{Spec}(R)}(bR)$ inside $\mathrm{Spec}(R)$ in the usual way, show that if X is a *finite dimensional* irreducible component of $\mathrm{Spec}(R)$ and Y is an irreducible component of $\mathrm{Spec}(S)$ with $Y \subseteq X$, then $\dim Y < \dim X$.

Points below; maximum score: 90; exam grade: score/10+1			
1a: 7 1b: 5	2a: 12 2b: 6 2c: 6	3a: 6 3b: 8 3c: 8 3d: 8	4a: 4 4b: 7 4c: 6 4d: 7

An additional practice problem

(1) (a) Let $R \subseteq S$ be an integral extension of rings. Show that the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is closed.

(b) Let k be a field, x, y variables, $B = k[x] \times k[y]$, and

$$A = \{(f(x), g(y)) \text{ in } B \text{ with } f(0) = g(0)\} \ .$$

(i) Show that B is integral over A.

(ii) Prove that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is *not* an open map. (Hint: consider $\mathcal{V}_{\mathrm{Spec}(B)}(k[x] \times \{0\})$ and its image in $\mathrm{Spec}(A)$.)

(c) Prove that the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open if *all* of the following hold:

- $R \subseteq S$ is an integral extension of rings,
- S is a domain,
- R is a Noetherian ring of Krull dimension 1.

(Hint: first classify the closed subsets of $\mathrm{Spec}(R)$.)