Mastermath Algebraic Geometry 1, Exam 2019/01/29

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- Time allowed: 3 hours.
- There are 4 questions, and 93 points. You get 7 points for free.
- You may quote results from the lecture notes without proof. If you wish to use results from the course exercises then you are expected to re-prove them.
- You may use previous parts of an exercise also when you have not proved them.
- Pen and paper only allowed - no books, notes, calculators etc.
- Throughout, k denotes an algebraically closed field, not necessarily of characteristic zero, and all varieties we consider are varieties over k.

1. (a) Let X be a topological space and $Y \subset X$ a subset, with the induced topology. Assume that Y is irreducible. Prove that its closure $\overline{Y} \subset X$, with the induced topology, is irreducible (9 points).

   (b) Let X be an affine variety and $P, Q \in X$ distinct. Prove that there exists an $f \in \mathcal{O}_X(X)$ such that $f(P) = 0$ and $f(Q) = 1$. Does the same hold when X is projective? *Hint: think in terms of polynomials, or in terms of ideals* (9 points).

2. Consider the map $\widetilde{\varphi}\colon \mathbb{A}^3 \to \mathbb{A}^6$, $\quad \widetilde{\varphi}(a,b,c) = (a^2, ab, b^2, ac, c^2, bc)$.

   (a) Show that $\widetilde{\varphi}$ induces a map $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^5$, $(a:b:c) \mapsto (a^2 : ab : b^2 : ac : c^2 : bc)$, and that this map φ is a morphism of varieties (6 points).

   (b) Let $S := \operatorname{im}\varphi$ (it is called the Veronese surface). Give a set T of homogeneous elements of $k[x_0, \ldots, x_5]$ such that $S = Z_{\mathrm{proj}}(T)$, and prove this equality. (12 points).

   (c) Prove that φ induces an isomorphism from $\mathbb{P}^2$ to S (6 points).

   (d) Let $C := Z_{\mathrm{proj}}(f) \subset \mathbb{P}^2$, where $f \in k[x,y,z]$ is homogeneous of degree 2 and $f \neq 0$. Prove that $\mathbb{P}^2 \setminus C$ is affine. *Hint: You may use that in any variety the intersection of an affine open subvariety with a closed subset is an affine variety* (6 points).

   (e) Let $D \subset S$ be an irreducible closed subset of dimension 1. Show that there is a homogeneous g in $k[x_0, \ldots, x_5]$ such that $D = Z_{\mathrm{proj}}(g) \cap S$. (6 points).

3. (a) Give the definitions of presheaf and sheaf of abelian groups on a topological space X (8 points).

   (b) Let $\varphi\colon X \to Y$ be a continuous map between topological spaces and let $\mathcal{F}$ be a presheaf of abelian groups on X. Let $\varphi_*\mathcal{F}$ be the presheaf on Y with, for any open subset $U \subset Y$, $(\varphi_*\mathcal{F})(U) := \mathcal{F}(\varphi^{-1}(U))$, and with the obvious restriction maps of $\mathcal{F}$. Prove that if $\mathcal{F}$ is a sheaf, then $\varphi_*\mathcal{F}$ is a sheaf (7 points).

4. Let C be an irreducible smooth projective curve. Recall from the course exercises that there exists a divisor K_C, determined up to linear equivalence, such that $\Omega^1_C \cong \mathcal{O}_C(K_C)$. In the following you may use that for any divisor D on C, $\dim H^0(C, \mathcal{O}_C(D)) = \dim H^1(C, \mathcal{O}_C(K_C - D))$.

   (a) Let g be the genus of C. Show that $\dim H^0(C, \Omega^1_C) = g$ and $\deg K_C = 2g - 2$ (12 points).

   (b) Now assume that $g > 0$, and let $P \in C$. Prove that there exists an ω in $\Omega^1_C(C)$ such that $\omega(P) \neq 0$. You may use that $H^0(C, \mathcal{O}_C(P)) = 1$. *Hint: determine the dimension of* $H^0(C, \mathcal{O}_C(K_C - P))$ (12 points).