Mastermath Algebraic Geometry 1, Retake 2019/03/05

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- Time allowed: 3 hours.
- There are 3 exercises, and 100 points.
- You may quote results from the lecture notes without proof. If you wish to use results from the course exercises then you are expected to re-prove them.
- You may use previous parts of an exercise also when you have not proved them.
- Pen and paper only allowed - no books, notes, calculators etc.
- Throughout, k denotes an algebraically closed field, not necessarily of characteristic zero, and all varieties we consider are varieties over k.

1. (a) Let X be an affine variety and $P, Q \in X$ distinct points. Show that there exist regular functions $f, g\colon X \to k$ such that $f(P) = g(Q) = 0$ and $f(R) + g(R) = 1$ for all $R \in X$ (7 points).

   (b) Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X and let $P \in X$. Give the definition of the stalk $\mathcal{F}_P$ (7 points).

   (c) Let X be an irreducible variety and $P \in X$. Show that there exists a well-defined injective ring homomorphism $\mathcal{O}_{X,P} \hookrightarrow K(X)$ (7 points).

2. An irreducible conic is an projective variety of the form $C := Z_{\mathrm{proj}}(f) \subset \mathbb{P}^2$, where $f \in k[x, y, z]$ is irreducible, non-zero, and homogeneous of degree 2.

   (a) Let $C \subset \mathbb{P}^2$ be an irreducible conic. Show that there exists an automorphism $T\colon \mathbb{P}^2 \to \mathbb{P}^2$, such that $T(C) = Z_{\mathrm{proj}}(xy + xz + yz)$. *Hint: show that there are three points $P, Q, R \in C$ which are not collinear.* (12 points)

   (b) Prove that $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^2$, $\varphi(a : b) = (a^2 : b^2 : ab)$ is a well-defined morphism (8 points).

   (c) Show that the image of φ is an irreducible conic and φ is an isomorphism onto its image (8 points).

   (d) Prove that any irreducible conic $C \subset \mathbb{P}^2$ is smooth (8 points).

   (e) Let $k = \mathbb{C}$. Given any irreducible conic $C \subset \mathbb{P}^2$ such that $(0 : 0 : 1) \notin C$. Consider the morphism $f\colon C \to \mathbb{P}^1$, $(x : y : z) \to (x : y)$. Using only the fact that $\mathbb{P}^1$ has genus 0 and the previous parts of this exercise, show that the Riemann-Hurwitz formula holds for the analytification of f. *Hint: Show that C is defined by $z^2+a(x,y)z+b(x,y)$ with a homogeneous of degree 1 and b homogeneous of degree 2. Then show that the discriminant $\Delta = a^2 - 4b$ is not the zero polynomial and has two distinct roots on $\mathbb{P}^1$.* (12 points)

3. In this exercise $k = \mathbb{C}$. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the zero set of the bi-homogeneous polynomial $f(x_0, x_1, y_0, y_1) = (x_0^3 + x_1^3)y_0^2 + (x_0^3 - x_1^3)y_1^2$. Consider the morphism $\varphi\colon C \to \mathbb{P}^1$, which sends $((a_0 : a_1), (b_0 : b_1))$ to $(a_0 : a_1)$.

   (a) State the definition of smoothness of a variety at a point (7 points).

   (b) Prove that C is a smooth curve (8 points).

   (c) Determine the degree of φ and use Riemann-Hurwitz to show that the genus of C is 2 (8 points).

   (d) Let $P \in C$. Show that any divisor D on C satisfies $\deg D \leq \dim H^0(C, \mathcal{O}_C(D + P))$ (8 points).