EXAMINER: H.T.C. STOOF	UTRECHT UNIVERSITY
DATE: 10/11/22	MIDTERM EXAM
TIME: 13:30 - 16:30	

Midterm exam for Statistical Field Theory

- Write your **name and student number on every sheet**.
- There are 3 problems. Write your answers to individual problems on different sheets.
- Make sure that your **answers are understandable and readable**. In doubt, explain with a short comment on what you are doing.

Problem 1: Energy of non-interacting and interacting electron spins. We consider two electrons (fermions) with spin 1/2 at two different lattice sites (positions). As spin-quantization axis, we choose the z-axis and the lattice sites are indexed by 1 and 2. So, for example, $\hat{\psi}^\dagger_{1,\uparrow}$ creates an electron at site 1 with spin $\uparrow$ in the z-direction and $\hat{\psi}_{2,\downarrow}$ annihilates an electron at site 2 with spin $\downarrow$ in the z-direction. In general $\hat{\psi}^\dagger_{i,\alpha}$ and $\hat{\psi}_{i,\alpha}$, respectively, create and annihilate an electron at site i with spin α in the z-direction. We assume, in first instance, that electrons cannot move from one lattice site to the other.

(i) **[5pt]** The spin-operator for site i is given by

$$\hat{\mathbf{S}}_i = \frac{\hbar}{2}\sum_{\alpha,\beta} \hat{\psi}^\dagger_{i,\alpha}\boldsymbol{\sigma}_{\alpha,\beta}\hat{\psi}_{i,\beta} \ , \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. Show that the operator of site i for the spin in the x-direction is given by

$$\hat{S}^x_i = \frac{\hbar}{2}(\hat{\psi}^\dagger_{i,\uparrow}\hat{\psi}_{i,\downarrow} + \hat{\psi}^\dagger_{i,\downarrow}\hat{\psi}_{i,\uparrow}) \ . \tag{2}$$

(ii) **[5pt]** If both electrons are exposed to a magnetic field in the x-direction, so $\mathbf{B} = (B, 0, 0)$, their Hamiltonian is given by

$$\hat{H}_B = -\mu B(\hat{S}^x_1 + \hat{S}^x_2) \ , \tag{3}$$

where μ is the magnetic moment of electrons. Determine first the normalization constants and then calculate the system's average energy for the three different states

$$|\psi_{f+}\rangle \propto (\hat{\psi}^\dagger_{1,\uparrow} + \hat{\psi}^\dagger_{1,\downarrow})(\hat{\psi}^\dagger_{2,\uparrow} + \hat{\psi}^\dagger_{2,\downarrow})\,|0\rangle \ , \tag{4}$$

$$|\psi_{f-}\rangle \propto (\hat{\psi}^\dagger_{1,\uparrow} - \hat{\psi}^\dagger_{1,\downarrow})(\hat{\psi}^\dagger_{2,\uparrow} - \hat{\psi}^\dagger_{2,\downarrow})\,|0\rangle \ , \tag{5}$$

$$|\psi_{a}\rangle \propto (\hat{\psi}^\dagger_{1,\uparrow} + \hat{\psi}^\dagger_{1,\downarrow})(\hat{\psi}^\dagger_{2,\uparrow} - \hat{\psi}^\dagger_{2,\downarrow})\,|0\rangle \ . \tag{6}$$

Note: the indices f and a are short for ferromagnetic and antiferromagnetic; however, this is not important to solve the problem.

(iii) **[5pt]** Now, instead of being exposed to a magnetic field, we assume that the electron spins are interacting with an interaction described by the Hamiltonian

$$\hat{H}_J = -J\hat{S}^x_1\hat{S}^x_2 \ , \tag{7}$$

where J is the exchange constant (a real number). Calculate the system's average energy for the states $|\psi_{f\pm}\rangle$ and $|\psi_a\rangle$ from above; see equations (4), (5) and (6).

(iv) **[5pt] BONUS** If the electrons could move from one lattice site to the other, could both electrons be at the same site? If yes, write down a corresponding state. If no, explain why it is not possible.

Problem 2: Bosonic and fermionic coherent states. We consider now a system with a boson in the single-particle state $\chi_B(\mathbf{x})$ of energy ϵ_B and a fermion in the single-particle state $\chi_F(\mathbf{x})$ of energy ϵ_F; that is, we consider a system with the Hamiltonian

$$\hat{H} = \epsilon_B \hat{\psi}_B^\dagger \hat{\psi}_B + \epsilon_F \hat{\psi}_F^\dagger \hat{\psi}_F \ , \tag{8}$$

where $\hat{\psi}^\dagger$ creates a particle in that state and $\hat{\psi}$ is the corresponding annihilation operator. Note that the index specifies, if $\hat{\psi}^\dagger$ and $\hat{\psi}$ are fermionic or bosonic creation and annihilation operators.

(i) [5pt] **BONUS** Depending on the particle type (bosons or fermions), is there a largest possible energy for a system described by H? If yes, give its value. If no, explain why not.

(ii) [10pt] Show that $|\phi_B, \phi_F\rangle = e^{\phi_B \hat{\psi}_B^\dagger - \phi_F \hat{\psi}_F^\dagger} |0\rangle$ is the coherent state for the single-particle states $\chi_{B,F}(\mathbf{x})$; that is, show that

$$\hat{\psi}_B |\phi_B, \phi_F\rangle = \phi_B |\phi_B, \phi_F\rangle \ , \tag{9}$$

$$\hat{\psi}_F |\phi_B, \phi_F\rangle = \phi_F |\phi_B, \phi_F\rangle \ . \tag{10}$$

Then, calculate the average energy of the coherent state

$$E(\phi_B^*, \phi_B, \phi_F^*, \phi_F) = \frac{\langle \phi_B, \phi_F | \hat{H} | \phi_B, \phi_F \rangle}{\langle \phi_B, \phi_F | \phi_B, \phi_F \rangle} \ . \tag{11}$$

What is the average number of bosons in the coherent state?

(iii) [5pt] Determine from your answer also the derivatives

$$\frac{\partial}{\partial \phi_F} E(\phi_B^*, \phi_B, \phi_F^*, \phi_F), \frac{\partial}{\partial \phi_F^*} E(\phi_B^*, \phi_B, \phi_F^*, \phi_F), \tag{12}$$

$$\frac{\partial^2}{\partial \phi_F^* \partial \phi_F} E(\phi_B^*, \phi_B, \phi_F^*, \phi_F) \ , \tag{13}$$

and the integral

$$\int d\phi_F^* d\phi_F E(\phi_B^*, \phi_B, \phi_F^*, \phi_F) \ . \tag{14}$$

Problem 3: Harmonic oscillator with coherent states. In this problem, we consider only one non-relativistic particle in a quadratic potential $V(\hat{x}) = m\omega^2\hat{x}^2/2 - \hbar\omega/2$. So, the Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - \frac{\hbar\omega}{2}, \tag{15}$$

where the mass m and the frequency ω are just real numbers but $\hat{x}$ and $\hat{p}$ are position and momentum operators with the commutation relation $[\hat{x}, \hat{p}] = i\hbar$.

(i) **[5pt]** Introducing the annihilation and creation operators as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \tag{16}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right), \tag{17}$$

show that $[\hat{a}, \hat{a}^\dagger] = 1$ and that the Hamiltonian can be written as

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}. \tag{18}$$

(ii) **[5pt]** Solve the time-dependent Schrödinger equation for the coherent state $|\alpha(t)\rangle$, that is, give the explicit time dependence of $\alpha(t)$, where $|\alpha(t)\rangle$ is the **normalized** coherent state of $\hat{a}$, so that $\hat{a}\,|\alpha(t)\rangle = \alpha(t)\,|\alpha(t)\rangle$.

(iii) **[5pt]** Determine $x_\alpha(t) = \langle\alpha(t)|\,\hat{x}\,|\alpha(t)\rangle$ and $p_\alpha(t) = \langle\alpha(t)|\,\hat{p}\,|\alpha(t)\rangle$ explicitly in terms of $\alpha(t)$. Based on your previous results, argue why the coherent state behaves as a classical state. For simplicity, you may assume that $\alpha(0)$ is real, which corresponds to a specific choice for the initial condition at $t = 0$.

Hint: To argue that the coherent state behaves as a classical state, you might want to compare $x_\alpha(t)$ and $p_\alpha(t)$ to Hamilton's classical equations of motion $dp/dt = -\partial_x H(x,p)$ and $dx/dt = \partial_p H(x,p)$, where $H(x,p) = p^2/2m + m\omega^2x^2/2 - \hbar\omega/2$ is the classical Hamiltonian.

(iv) **[5pt] BONUS** What is the essential difference between creation and annihilation operators of the harmonic oscillator on the one hand and the creation and annihilation operator of second quantization on the other hand?

Hint: You might want to think about particle numbers.