Department of Information and Computing Sciences, Faculty of Science, UU.
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In 2010-2011, the course INFOAN was given by Dr. H.L. Bodlaender.

2nd exam Algorithms and Networks (INFOAN)
April 18, 2011

You may give your answers in Dutch or in English. Write clearly. You may consult four sides of A4 with notes. Results used in the course or exercise sets may be used without further proof, unless explicitly asked.
Some parts are harder than others: use your time well, and make sure you first finish the easier parts!
Good luck!

Question 1. Sorting in logarithmic space (2 points)

Explain the main ideas of the algorithm that sorts a list of elements using *logarithmic space*.
You may use at most 20 lines of text.

Question 2. NP-completeness (2 points)

Show that the following problem is NP-complete:

`DIRECTED FEEDBACK VERTEX SET`
Given: Directed graph $G = (V, E)$, integer $K \leq |V|$
Question: Is there a set $W \subseteq V$ with $|W| \leq K$, such that each cycle in G contains at least one vertex in W?

(Hint: use the `VERTEX COVER` problem.)

Question 3. Trees and counting independent sets (2 points)

Consider a rooted tree $T = (V, E)$. For each vertex $v \in V$, we let $T_v = (V_v, E_v)$ be the subtree of T, consisting of v and all its descendants in T, and all edges between these vertices. I.e., V_v denotes the set consisting of v and all its descendants.
We want to compute how many different independent sets T has. For instance, if T has two vertices v, w and one edge $\{v, w\}$, then T has three independent sets: $\{v\}$, $\{w\}$, and $\emptyset$.
Define:

- $X(v)$ is the number of independent sets in T_v.
- $Y(v)$ is the number of independent sets $W \subseteq V_v$, in T_v, with $v \notin W$.

a) Suppose v is a leaf in T. Explain why $X(v) = 2$. What is $Y(v)$?

b) Suppose v has children $w_1, \ldots, w_r, r \geq 1$. Give a clear explanation why the following formula holds:
$$X(v) = Y(w_1) \cdot Y(w_2) \cdots Y(w_r) + X(w_1) \cdot X(w_2) \cdots X(w_r)$$

c) Suppose v has children $w_l, \ldots, w_r, r \geq 1$. Give a formula that expresses $Y(v)$ with help of (some or all of) the values $X(w_1), \ldots, X(w_r), Y(w_1), \ldots, Y(w_r)$.

d) Give an algorithm that computes in linear time the number of independent sets of a given tree. (Note: the input to the algorithm is a tree, the output is an integer.)

Question 4. A kernel for Cluster Editing with Size Bound (2 points)

Consider the following problem:

Given: Undirected graph $G = (V, E)$, integer K
Parameter: K
Question: Can we make at most K changes to G, such that we obtain a graph where each connected component is a clique with at most 7 vertices; where each change is either the addition of an edge, or the deletion of an edge.

I.e., we are given a graph and an integer K, and want to make at most K modifications to the graph: a modification is either the addition of an edge or the deletion of an edge.
In this exercise, we show that this problem has a kernel with $O(K)$ vertices.

a) Suppose G has more than one connected component. Argue that we never need to add edges between vertices in different connected components in an optimal solution.

b) Suppose G has a connected component that is a clique with at most seven vertices. Give a safe rule that transforms your input to a smaller equivalent one.

c) Suppose G has a connected component with at least r vertices with $r \geq 8$. Show that we need at least $\lfloor r/14 \rfloor$ changes that involve the addition or removal of edges between vertices in this component.

d) Show that the problem has a kernel with $O(K)$ vertices. *Describe your algorithm clearly, and describe clearly that it gives the desired bound.*
(If you cannot get an $O(K)$ bound: you get part of the points if you get a kernel with a polynomial bound.)

Question 5. Branching algorithm for Cluster Vertex Deletion with Size Bound (2 points: 0.5+0.5+1)

Consider the following parameterized problem:

Given: Undirected graph $G = (V, E)$, integer K
Parameter: K
Question: Is there a set $W \subseteq V$ of G with $|W| \leq K$, such that if we delete all vertices in W and their incident edges from G, then each connected component in G is a clique with at most seven vertices.

a) Suppose G contains three vertices v, w and x, with $\{v, w\} \in E$, $\{v, x\} inE$, and $\{w, x\} \notin E$. Show that in any optimal solution, $v \in W$ or $w \in W$ or $x \in W$.

b) What form has a graph G that does not contain any set of three vertices v, w and x, with $\{v, w\} \in E$, $\{v, x\} inE$, and $\{w, x\} \notin E$?

c) Show that the problem belongs to FPT.