Exam Algorithms and Networks 2013/2014

You have three hours for the exam. You may give your answers in Dutch or in English. Write clearly. You may consult four sides of A4 with notes.

Results described in the course or exercises may be used without further proof, unless explicitly asked. This includes that you may use the NP-completeness of all problems shown or discussed to be NP-complete in the course or exercise sets.

Some parts are harder than others: use your time well, and make sure you first finish the easier parts!

Good luck!

The number of vertices of a graph $G = (V, E)$ is denoted by n.

1. NP-completeness and graph problems: 0.5 + 0.5 + 0.5 point. Consider the following problem.

> [2-COLORABILITY]
> **Given:** Undirected, connected graph $G = (V, E)$.
> **Question:** Is there a function $c : V \to \{1, 2\}$, such that for all $\{v, w\} \in E$, $c(v) \neq c(w)$?

(a) Show that this problem belongs to the class P.
(b) If $P \neq NP$, does this problem belong to the class NP? Explain briefly.
(c) If $P \neq NP$, is this problem NP-complete? Explain briefly.

2. NP-completeness: Edge Coverage: 1.5 point. Consider the following problem.

> [EDGE COVERAGE]
> **Given:** Undirected graph $G = (V, E)$, integer $k \leq |V|$, integer $q \leq |E|$.
> **Question:** Is there a set of at most k vertices that covers at least q edges, i.e., is there a set $W \subseteq V$ with $|W| = k$, such that $|\{\{v, w\} \in E \mid v \in W \vee w \in W\}| \geq q$?

Prove that this problem is NP-complete.

3. Fixed parameter tractability. 1.5 point. Consider following problem.

> [Dominating Set on Graphs of Maximum Degree 5]
> **Given:** Undirected graph $G = (V, E)$ such that each vertex in V has at most degree five; integer $K \leq |V|$.
> **Question:** Does G have a dominating set of size at most K, i.e., is there a set $W \subseteq V$, with $|W| \leq K$ and for each $v \in V$, we have $v \in W$ or $\exists w \in W$ with $\{v, w\} \in E$?

Show that this problem belongs to the class FPT.

Remark: there is more than one correct solution to this question.

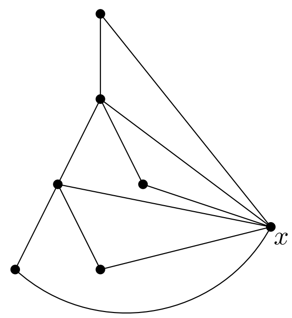
4. Graph Isomorphism for cluster graphs. 1.5 point. An undirected graph $G = (V, E)$ is a *cluster graph*, if each connected component of G is a clique.

Give an algorithm that solves the Graph Isomorphism problem on cluster graphs in polynomial time.

You do not need to prove correctness of your algorithm.

5. Treewidth. 1 point. Suppose we have an undirected graph $G = (V, E)$, made in the following way. First, we take a tree $T = (V', E')$. Then, we add one additional vertex x, and make x adjacent to all vertices in V' (i.e., x is a *universal* vertex). An example is given in the figure below.

Show that such a graph G has treewidth at most 2.



6. Dynamic programming on trees. 1.5 point. In the following problem, we consider the problem to find the minimum cost of a 3-coloring of a tree of maximum degree three. Given is a tree such that each vertex has at most three neighbors. A 3-coloring is a function that maps to each vertex a number from $\{1, 2, 3\}$, such that adjacent vertices have different numbers assigned to it. The cost of a 3-coloring is the sum over all numbers that are given.

I.e., we consider the following problem.

> MIN COST 3-COLORING ON TREES
> **Given:** A tree $T = (V, E)$ with maximum degree three
> **Question:** What is the minimum cost of a 3-coloring of T, i.e., what is the minimum of $\sum_{v \in V} f(v)$ over all functions $f : V \to \{1, 2, 3\}$ with for all $\{v, w\} \in E$: $f(v) \neq f(w)$.

Show that this problem can be solved in linear time.

7. NP-completeness: Hamiltonian Path. 1.5 point.

> [HAMILTONIAN PATH WITH SPECIFIED ENDPOINTS]
> **Given:** Graph $G = (V, E)$, vertices $s, t \in V$.
> **Question:** Is there a path in G from s to t that visits each vertex in V exactly once?

Prove that this problem is NP-complete.