GALOIS THEORY - EXAM B (24/06/2010)

- On each sheet of paper you hand in write your name and student number
- Do not provide just final answers. Prove and motivate your arguments!
- The use of computer, calculator, lecture notes, or books is not allowed
- Each problem is worth 25 points. A perfect solution to a complete problem gives a bonus of 0.2 points. The final mark is the minimum between the points earned and 10.

Problem A

(1) Let K be a field and $L : K$ a finite extension. Prove that $|Gal(L : K)| \leq [L : K]$.

(2) Consider the field $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11})$. Let $\alpha_1, \cdots, \alpha_{32}$ be the elements of the form $\pm\sqrt{2} \pm \sqrt{3} \pm \sqrt{5} \pm \sqrt{7} \pm \sqrt{11}$ (each choice for signs gives one of the 32 elements). Let $\alpha_{33} = (\sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{7} + \sqrt{11})^{13}$. Prove that there is no field automorphism $\sigma : L \to L$ that satisfies $\sigma(\alpha_i) = \alpha_{i+1}$ for all $1 \leq i < 33$ and $\sigma(\alpha_{33}) = \alpha_1$.

Problem B Let $L = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$

(1) Calculate $[L : \mathbb{Q}]$

(2) Prove that $\mathbb{Q}(\sqrt{2}) \subseteq L$

(3) Prove that $L : K$ is a Galois extension (hint: $\sqrt{2 + \sqrt{2}} \cdot \sqrt{2 - \sqrt{2}} = \sqrt{2}$)

(4) Prove that $Gal(L : \mathbb{Q})$ is cyclic

Problem C Let K be a field and $f(x) \in K[X]$ a separable polynomial which factorises in $K[X]$ as $f(x) = g(x)h(x)$. Let L_f, L_g and L_h be splitting fields of (respectively) f, g, and h over K.

(1) Show that the splitting fields can be chosen so that $L_g \subseteq L_f$ and $L_h \subseteq L_f$. From now on we assume such a choice was made and let $G_f = Gal(L_f : K)$, $G_g = Gal(L_f : L_g)$, and $G_h = Gal(L_f : L_h)$.

(2) Prove that G_g and G_h are normal subgroups of G_f.

(3) Prove that $G_g \cap G_h = \{1\}$.

(4) Prove that if $L_g \cap L_h = K$ then $G_g \cdot G_h = G_f$.

Problem D For each of the following statements decide if it is true or false and give a short argument to support your answer.

(1) Every field of characteristic 0 can be embedded in $\mathbb{C}$.

(2) Let $f(x) \in \mathbb{Q}[X]$ be a polynomial of degree n and L a splitting field of f over $\mathbb{Q}$. If $[L : \mathbb{Q}] = n!$ then f is irreducible.

(3) Let $\mathbb{F}_{p^n}$ be a field with p^n elements (p a prime number). Then for any $1 \leq m \leq n$ there is a subfield $G_m \subseteq \mathbb{F}_{p^n}$ with $|G_m| = p^m$.

(4) Let $\alpha \in \mathbb{R}$ be such that $\mathbb{Q}(\alpha) : \mathbb{Q}$ is Galois. If $[\mathbb{Q}(\alpha) : \mathbb{Q})] = 2^{22}$ then α is constructible.