GALOIS THEORY - RETAKE EXAM B (26/08/2010)

- On each sheet of paper you hand in write your name and student number
- Do not provide just final answers. Prove and motivate your arguments!
- The use of computer, calculator, lecture notes, or books is not allowed
- Each problem is worth 2.5 points. A perfect solution to a complete problem gives a bonus of 0.2 points. The final mark is the minimum between the points earned and 10.

Problem A

(1) Let K be a field and $L : K$ a finite extension. Prove that $|Gal(L : K)| \leq [L : K]$.

(2) Let $L : \mathbb{Q}$ be a field extension. Suppose that for every $n \in \mathbb{N}$ with $n > 1$ and any n different elements $a_1, \cdots, a_n \in L - \mathbb{Q}$ there exists a field automorphism $\sigma : L \to L$ such that for all $1 \leq i < n$ holds $\sigma(a_i) = a_{i+1}$ and $\sigma(a_n) = a_1$. Prove that $L : \mathbb{Q}$ is **not** finite.

Problem B Let L be the splitting field of $X^3 - 2$ over $\mathbb{Q}$.

(1) Compute $Gal(L : \mathbb{Q})$. Indicate its size and specify if it is cyclic or not. Give explicit generators for it and indicate the relations they satisfy.

(2) Find all intermidiate fields $\mathbb{Q} \subseteq M \subseteq L$ such that $M : \mathbb{Q}$ is normal. For each such M give an explicit description of the elements in it.

Problem C Let K be a field with $char(K) = 0$ and $f \in K[X]$ an irreducible polynomial. Let L be a splitting field of f over K. Prove that if $Gal(L : K)$ is abelian then $[L : K] = deg(f)$.

Problem D For each of the following statements decide if it is true or false and give a short argument to support your answer.

(1) Let $L : K$ and $L' : K$ be two finite extensions. If there exists a field isomorphism $\sigma : L \to L'$ such that $\sigma|_K = id$ then $Gal(L : K) \cong Gal(L' : K)$.

(2) Let $f(x) \in \mathbb{Q}[X]$ be a polynomial of degree n. If L is a splitting field of f over $\mathbb{Q}$ then $[L : \mathbb{Q}] \leq n!$.

(3) Every finite field is a splitting field, over its base field, of some polynomial.

(4) Let $\alpha \in \mathbb{R}$. if $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ then α is constructible.