TENTAMEN A RINGEN EN GALOISTHEORIE 22-04-2010

- On each sheet of paper you hand in write your name and student number
- Do not provide just final answers. Prove and motivate your arguments!
- The use of computer, calculator, lecture notes, or books is not allowed

Problem A (20 points) Consider the set $\mathbb{Q}_k = \{\frac{m}{n} \in \mathbb{Q} \mid n \neq 0(mod\, k)\}$, where $k \in \mathbb{N}$.

(1) Prove that $\mathbb{Q}_k$ is a sub-ring of $\mathbb{Q}$ if, and only if, k is a prime number.
(2) Let p be a prime number. Prove that (p) is a prime ideal in $\mathbb{Q}_p$.
(3) Prove that $\mathbb{Q}_p/(p)$ is a field.

Problem B (20 points) Let $P(X) = X^4 - X^3 + X^2 - X + 1$.

(1) Prove that $P(X)$ is reducible in $\mathbb{Z}_5[X]$.
(2) Factorize $P(X)$ in $\mathbb{Z}[X]$ and in $\mathbb{Q}[X]$.
(3) Find a polynomial $Q(X)$ that is irreducible over $\mathbb{Q}$ but is reducible over the field $\mathbb{F}_2$ with two elements.

Problem C (20 points) Let $L : K$ be an algebraic field extension and $\alpha \in L$.

(1) Prove that if $[K(\alpha) : K]$ is odd then $K(\alpha) = K(\alpha^2)$.
(2) Show that the converse does not hold by exhibiting an algebraic extension $L : K$ and $\alpha \in L$ such that $K(\alpha) = K(\alpha^2)$ holds yet $[K(\alpha) : K]$ is even.

Problem D (60 points) For each of the following statements decide if it is true or false. Give a short argument to support your answer.

(1) The ring $\mathbb{Z}[\sqrt{-5}]$ is a UFD (Unique Factorization Domain)
(2) $\mathbb{Q}[X]/(X^2 - 1) \cong \mathbb{Q} \times \mathbb{Q}$
(3) Let K be a field and $p \in \mathbb{N}$, $p > 1$. If the function $\psi : K \to K$ given by $\psi(x) = x^p$ is a ring homomorphism with $\psi(1) = 1$ then $char(K) \neq 0$.
(4) Let R be a domain and $r \in R$. If r is prime then r is irreducible.
(5) There exists a number $x \in \mathbb{Z}$ such $\mathbb{Z}/(x)$ is a domain but not a field.
(6) $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean ring.
(7) The polynomial ring $\mathbb{C}[X, Y, Z]$ is a UFD (Unique Factorization Domain)
(8) The polynomial $X^{17} + 4X^{10} - 2X^2 - 18$ is irreducible in $\mathbb{Q}[X]$.
(9) Let A be the set of all complex numbers that are algebraic over $\mathbb{Q}$. Then the field of fractions of A is isomorphic to $\mathbb{C}$.
(10) If R be a commutative ring then the set of all nilpotent elements in R forms an ideal.